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Kévin Santugini-Repiquet

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Abstract

In this paper, we study the homogenization of the demagnetization field operator in periodically perforated domains using the two-scale convergence method. As an application, we homogenize the Landau-Lifshitz equation in such domains. We consider regular homothetic holes.

1 Introduction

Due to their properties, ferromagnetic materials are nowadays widely used in the industry. In particular, nonhomogenous periodic ferromagnetic configurations are the subject of a growing interest: these periodic configurations may exhibit properties difficult to achieve with homogenous materials. To correctly predict the magnetic behavior of these configurations is of prime importance. As the period length decreases, the cost of direct numerical simulations increases and become prohibitive. A more practical approach would first involve the use of homogenization: homogenization without holes dates back from Bensoussan et al [4]. In perforated domains, it has been studied by Cioranescu et al in [6] for local operators.

In the framework of the micromagnetic model of Brown [5] in the magnetostatic approximation, a global operator is present: the demagnetization field operator. This operator was homogenized in the case of multilayers in Hamdache [12]: besides the appearance of an intuitive factor corresponding to the quasi constant ratio of ferromagnetic material, also appears a purely local corrector term.

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The main goal of this paper is to homogenize the nonlocal demagnetization field operator term in a periodically perforated geometry and derive the local corrector term. First, we introduce some notations and define the perforated geometry in section 2. In section 3, we derive our main result: the homogenized demagnetization field operator for perforated domains. Finally, in section 4, we apply our main result and homogenize the nonlinear PDE governing the behavior of ferromagnetic material: the Landau-Lifshitz equation.

2 Notations and known results

2.1 Some function spaces

Given a measurable set $\mathcal{O}$, and a real number $p \geq 1$, we denote by $\mathrm{L}^p(\mathcal{O})$ the set of all measurable functions such that $\int_{\mathcal{O}} |u|^p \,\mathrm{d}\boldsymbol{x} < +\infty$. This is a Banach space for the norm

$$\|u\|_{\mathrm{L}^p(\mathcal{O})} = \left(\int_{\mathcal{O}} |u|^p \,\mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}}.$$

Given an open set $\mathcal{O}$ in $\mathbb{R}^n$, we denote by $\mathcal{D}'(\mathcal{O})$ the standard space of distributions over $\mathcal{O}$. If m belongs to $\mathbb{N}$ and $p \geq 1$, we denote by $\mathrm{W}^{m,p}(\mathcal{O})$ the subset of $\mathcal{D}'(\mathcal{O})$ containing all distributions u such that, for all multi-indices α, $|\alpha| \leq m$, $\mathrm{D}^\alpha u$ belongs to $\mathrm{L}^p(\mathcal{O})$. This is a Banach space for the norm:

$$\|u\|_{\mathrm{W}^{m,p}(\mathcal{O})} = \Big(\sum_{|\alpha| \leq m} \| \mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(\mathcal{O})} \Big)^{\frac{1}{p}}.$$

We set $\mathrm{H}^m(\mathcal{O}) = \mathrm{W}^{m,2}(\mathcal{O})$, this is an Hilbert space. We also set $\mathbb{L}^p(\mathcal{O}) = (\mathrm{L}^p(\mathcal{O}))^3$, $\mathbb{W}^{m,p}(\mathcal{O}) = (\mathrm{W}^{m,p}(\mathcal{O}))^3$, $\mathbb{H}^m(\mathcal{O}) = (\mathrm{H}^m(\mathcal{O}))^3$

We set $\mathcal{Y} = (0,1)^3$, and denote by $\mathcal{C}^\infty_\#(\mathcal{Y})$ the set of infinitely differentiable real functions over $\mathbb{R}^3$ that are 1-periodic on each of the three space variables. We define $\mathrm{H}^1_\#(\mathcal{Y})$ as the closure of $\mathcal{C}^\infty_\#(\mathcal{Y})$ in $\mathrm{H}^1(\mathcal{Y})$. By $\mathcal{C}^\infty(\overline{\Omega}) \otimes \mathcal{C}^\infty_\#(\mathcal{Y})$, we denote the set containing all infinitely differentiable real functions over $\overline{\Omega} \times \mathbb{R}^3$ that are 1-periodic on the three last variables. We define $\mathrm{H}^1_\#(\Omega \times \mathcal{Y})$ as the closure of $\mathcal{C}^\infty(\Omega) \otimes \mathcal{C}^\infty_\#(\mathcal{Y})$ in $\mathrm{H}^1(\mathcal{Y})$.

Finally, in this paper $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the canonical base of $\mathbb{R}^3$.

2.2 Two-scale convergence

In this section, we recall briefly the concept of two-scale convergence. For details and proofs of this subsection, we refer the reader to Allaire [1]. First, we recall the concept of acceptable functions and reproduce Definition 1.5 of [1]:

Definition 1. Given an open set $\mathcal{O}$, a function ϕ in $\mathrm{L}^2(\mathcal{O}\times(\mathbb{R}^3/\mathcal{Y}))$, is said to be acceptable if

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}}\left|\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon}\right)\right|^2\,\mathrm{d}\boldsymbol{x}=\int_{\mathcal{O}\times\mathcal{Y}}\left|\phi\left(\boldsymbol{x},\boldsymbol{y}\right)\right|^2\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

It has been shown in [1] that a function belonging to either $\mathrm{L}^2(\mathcal{O};\mathcal{C}_\#(\mathcal{Y}))$, $\mathrm{L}^2(\mathcal{Y};\mathcal{C}(\mathcal{O}))$ or $\mathcal{C}(\mathcal{O})\otimes\mathcal{C}_\#(\mathcal{Y})$ is acceptable.

Definition 2. Given an open set $\mathcal{O}$, let E be a subspace of $\mathrm{L}^2(\mathcal{O}\times(\mathbb{R}^3/\mathcal{Y}))$ such that every function in E is acceptable. A bounded sequence u_ε in $\mathrm{L}^2(\mathcal{O})$ is said to E-two-scale converge to u_0 belonging to $\mathrm{L}^2(\mathcal{O}\times\mathcal{Y})$, if for all ϕ in E:

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}}u_\varepsilon(\boldsymbol{x})\phi(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\,\mathrm{d}\boldsymbol{x}=\int_{\mathcal{O}}\int_{\mathcal{Y}}u_0(\boldsymbol{x})\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

In [1], it is shown that the two-scale convergence concept is the same for E among $\mathrm{L}^2(\mathcal{O};\mathcal{C}_\#(\mathcal{Y}))$, $\mathrm{L}^2(\mathcal{Y};\mathcal{C}(\mathcal{O}))$ and $\mathcal{C}(\mathcal{O})\otimes\mathcal{C}_\#(\mathcal{Y})$. Thereafter, in this article, two-scale convergence will always denote E-two-scale convergence with regards to any of the three previous choice of E.

We reproduce Theorem 1.2 of [1]:

Theorem 3. *Let u_ϵ be a bounded sequence of elements bounded in $\mathrm{L}^2(\mathcal{O})$, then there exists a subsequence $(\epsilon_k)_{k\in\mathbb{N}}$, and u_0 in $\mathrm{L}^2(\mathcal{O}\times\mathcal{Y})$ such that u_{ϵ_k} two-scale converges to u_0.*

Finally, we recall a simple criteria that justify the convergence of products: Theorem 1.8 of [1].

Theorem 4. *If u_ϵ and v_ϵ are bounded sequence in $\mathrm{L}^2(\mathcal{O})$ that respectively two-scale converge to u_0 and v_0 in $\mathrm{L}^2(\mathcal{O}\times\mathcal{Y})$, and if*

$$\|u_0\|_{\mathrm{L}^2(\mathcal{O}\times\mathcal{Y})}=\liminf_{\epsilon\to 0}\|u_\epsilon\|_{\mathrm{L}^2(\mathcal{O})},$$

then

$$\lim_{\epsilon\to 0}\int_{\mathcal{O}}u_\epsilon(\boldsymbol{x})v_\epsilon(\boldsymbol{x})\phi(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon})\,\mathrm{d}\boldsymbol{x}=\int_{\mathcal{O}}\int_{\mathcal{Y}}u_0(\boldsymbol{x},\boldsymbol{y})v_0(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x},$$

for all ϕ in $\mathcal{C}^\infty(\overline{\mathcal{O}})\otimes\mathcal{C}^\infty_\#(\mathcal{Y})$.

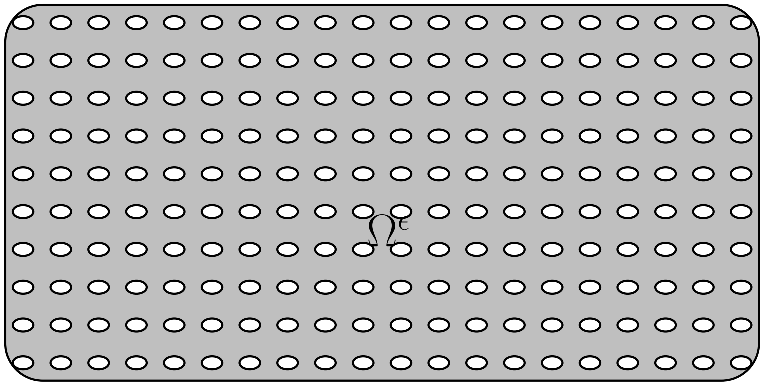

Figure 1: Example of perforated domain

2.3 The perforated geometry

In this section, we define unambiguously the regularly perforated geometries. Let

- Ω be an open bounded set of $\mathbb{R}^3$, with a smooth boundary,
- $\mathcal{T}_0 \subset \mathcal{Y}$ be a compact with a smooth boundary such that $\overline{\mathcal{T}_0^\circ} = \mathcal{T}_0$,
- $\mathcal{Y}^* = \mathcal{Y} \setminus \mathcal{T}_0$.

We set

$$T_\varepsilon = \bigcup_{\substack{k\in\mathbb{Z} \\ \varepsilon(k+\mathcal{Y})\subset\Omega}} \varepsilon(k + \mathcal{T}_0), \qquad\qquad \Omega^\varepsilon = \Omega \setminus T_\varepsilon. \tag{2.1}$$

An example of a possible Ω^ε can be seen in figure 1. In this paper, χ_ε is the characteristic function of Ω^ε. By $\chi_{\mathcal{Y}^*}$, we denote the characteristic function of $\mathcal{Y}^*$. We also set $\bar{\chi} = \int_{\mathcal{Y}} \chi_{\mathcal{Y}^*}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$. We also define the mean operator $⨍$ by

$$⨍_{\mathcal{O}} f\,\mathrm{d}\boldsymbol{x} = \frac{1}{|\mathcal{O}|}\int f\,\mathrm{d}\boldsymbol{x}.$$

To take the limit in integrals over perforated domains, we use Theorem 4 and the following lemma:

Lemma 5. *The sequence χ_ε two-scale converges to $\chi_{\mathcal{Y}^*}$. Moreover,*

$$\lim_{\varepsilon\to 0}\|\chi_\varepsilon\|_{\mathrm{L}^2(\Omega)} = \|\chi_{\mathcal{Y}^*}\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})}.$$

Equivalent results for $\mathbf{K}$ and $\mathbf{A}$ allow to take the two-scale limit in products involving $\mathbf{K}(\cdot,\cdot/\varepsilon)$ or $\mathbf{A}(\cdot,\cdot/\varepsilon)$.

2.4 Acceptable sequence of holes

We recall the concept of acceptable sequence of holes, see Damlamian et al [7]. It allows to homogenize nonlinear equations in perforated domains. In particular, it will be used for section 4.

Definition 6. The sequence of holes T_ε is acceptable if

(1) Any weak-$*$ limit of χ_ε is positive almost everywhere on Ω.

(2) There exists $c > 0$, independent of ε and a sequence of linear extension operators (P_ε) such that, for all $\varepsilon > 0$

$$P_\varepsilon \in \mathcal{L}(\mathrm{H}^1(\Omega^\varepsilon); \mathrm{H}^1(\Omega)),$$
$$(P_\varepsilon(v))_{|\Omega^\varepsilon} = v \qquad \forall v \in \mathrm{H}^1(\Omega^\varepsilon),$$
$$\|P_\varepsilon(v)\|_{\mathrm{H}^1(\Omega)} \leq c\|v\|_{\mathrm{H}^1(\Omega^\varepsilon)}.$$

Remark 7. Our concept of acceptable sequence of holes differs from the one of [7]. This is necessary because we study systems with Neumann boundary conditions on $\partial\Omega$ while Dirichlet conditions were considered in [7]. This is why T_ε is defined by (2.1) and not by $T_\varepsilon = \bigcup_{\substack{k\in\mathbb{Z} \\ \varepsilon(k+\mathcal{T}_0)\subset\Omega}} \varepsilon(k + \mathcal{T}_0)$ as in [7].

To homogenize partial differential equations in perforated domains, we need some extensions operators:

Lemma 8. *There exists $c > 0$ such that, for all $\varepsilon > 0$, there exists an extension operator P_ε from $\mathrm{H}^1(\Omega^\varepsilon)$ to $\mathrm{H}^1(\Omega)$ such that,*

$$P_\varepsilon u = u \qquad in\ \Omega^\varepsilon, \qquad\qquad \|P_\varepsilon u\|_{\mathrm{H}^1(\Omega)} \leq \|u\|_{\mathrm{H}^1(\Omega^\varepsilon)},$$

for all u in $\mathrm{H}^1(\Omega^\varepsilon)$.

Proof. The proof is similar to the proof of Proposition 1.8 in [7]. There exists a linear continuous extension operator P from $\mathrm{H}^1(\mathcal{Y}^*)$ to $\mathrm{H}^1(\mathcal{Y})$. We can then construct P_ε through scaling. □

2.5 The demagnetization field operator

The demagnetization field operator is the linear operator arising from the magnetostatic equations. It has been extensively studied by Friedman in [8, 9, 10]. We recall in this subsection its definition and its most important properties. In this paper, this operator is denoted by $\mathcal{H}_d$ and by definition sends any vector field $\boldsymbol{m}$ over the vector field $\boldsymbol{h}_d$ solution to

$$\operatorname{div}(\boldsymbol{h}_d) = -\operatorname{div}(\boldsymbol{m}), \qquad\qquad \operatorname{curl}(\boldsymbol{h}_d) = 0, \tag{2.2}$$

in the sense of distributions over the whole space $\mathbb{R}^3$. By Fourier transform, for $\boldsymbol{m}$ in $\mathcal{S}'(\mathbb{R}^3)$

$$\hat{\boldsymbol{h}}_d(\boldsymbol{\xi}) = -\frac{\boldsymbol{\xi}(\boldsymbol{\xi}\cdot\hat{\boldsymbol{m}})}{|\boldsymbol{\xi}|^2} + a\delta.$$

Therefore, up to an additive constant, the solution to system (2.2) in $\mathcal{S}'(\mathbb{R}^3)$ is unique. In this paper, we always consider $\boldsymbol{m} \in \mathbb{L}^2(\mathbb{R}^3)$ and require $\boldsymbol{h}_d \in \mathbb{L}^2(\mathbb{R}^3)$, therefore requiring $a = 0$. We have

$$\|\boldsymbol{h}_d\|_{\mathbb{L}^2(\mathbb{R}^3)} \leq \|\boldsymbol{m}\|_{\mathbb{L}^2(\mathbb{R}^3)}.$$

One verifies that $\boldsymbol{h}_d$ may be expressed using the kernel of the Laplace operator:

$$\boldsymbol{h}_d = -\nabla\left(\operatorname{div}(G * \boldsymbol{m})\right) = -\nabla(G * \operatorname{div}(\boldsymbol{m})), \tag{2.3a}$$

where

$$G = -\frac{1}{4\pi}\frac{1}{|x|}. \tag{2.3b}$$

We also introduce the concept of potential:

Definition 9. Let $\boldsymbol{m}$ be in $\mathbb{L}^2(\mathbb{R}^3)$. By its potential $\varphi(\boldsymbol{m})$, we denote the only solution in

$$W_0^1(\mathbb{R}^3) = \{u, \nabla u \in \mathbb{L}^2(\mathbb{R}^3), (\sqrt{1+r^2})u \in \mathrm{L}^2(\mathbb{R}^3)\},$$

to

$$\triangle\,\varphi = -\operatorname{div}\boldsymbol{m}.$$

The potential exists, see [3]. Obviously, $\mathcal{H}_d(\boldsymbol{m}) = \nabla\varphi(\boldsymbol{m})$.

3 Homogenization of the demagnetization field operator in perforated domain

In this section, we are interested in the two-scale limit of the demagnetization field operator as defined in the previous section. Let $\boldsymbol{m}_\varepsilon$ in $\mathbb{L}^2(\Omega^\varepsilon)$ two-scale converges to $\boldsymbol{m}_0$. Can we compute the two-scale limit of $\mathcal{H}_d(\boldsymbol{m}_\varepsilon)$? The answer is positive. The most straightforward way to compute this limit is to use the potential, see Definition 9. For the special case of multilayers, the computation was done by K. Hamdache [12].

First, we introduce the cell equation whose solutions are used to express the two-scale limit of the demagnetization field.

Definition 10. Let, for all $1 \le i \le 3$, w'_i be the unique solution in $\mathrm{H}^1_\#(\mathcal{Y})$ to

$$\int_{\mathcal{Y}} (\nabla_{\boldsymbol{y}} w'_i(\boldsymbol{x},\boldsymbol{y}) + \chi_{\mathcal{Y}^*}(\boldsymbol{y})\boldsymbol{e}_i)\, \nabla_{\boldsymbol{y}}\psi \,\mathrm{d}\boldsymbol{y} = 0,$$

$$⨍_{\mathcal{Y}} w'_i(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0,$$

for all ψ in $\mathrm{H}^1_\#(\mathcal{Y})$. By $\boldsymbol{w}'$, we denote the horizontal vector $[w'_1, w'_2, w'_3]$.

We then state our main result:

Proposition 11. *Let $\boldsymbol{u}^\varepsilon$ be a bounded sequence in $\mathbb{L}^2(\mathbb{R}^3)$ that two-scale converges to $\boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y})$. Then, the two-scale limit of $\boldsymbol{h}_d^\varepsilon = \mathcal{H}_d(\boldsymbol{u}^\varepsilon)$ is*

$$\boldsymbol{h}_d^0(\boldsymbol{x},\boldsymbol{y}) = \mathcal{H}_d(\int_{\mathcal{Y}} \boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}) + \nabla_{\boldsymbol{y}}\varphi_1^0(\boldsymbol{x},\boldsymbol{y}),$$

where $\varphi_1(\boldsymbol{x},\cdot)$ is the unique solution in $\mathrm{H}^1_\#(\mathcal{Y})$ to

$$\int_{\mathcal{Y}} \big(\boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}}\varphi_1^0(\boldsymbol{x},\boldsymbol{y})\big)\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0, \qquad \int_{\mathcal{Y}} \varphi_1^0 = 0,$$

for all ψ in $\mathrm{H}^1_\#(\mathcal{Y})$. Moreover, if

$$\boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \boldsymbol{u}^0(\boldsymbol{x}) & \text{if } \boldsymbol{y} \in \mathcal{Y}^*, \\ 0 & \text{if } \boldsymbol{y} \in \mathcal{T}_0, \end{cases}$$

then

$$\varphi_1^0(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{3} (\boldsymbol{u}^0(\boldsymbol{x})\cdot\boldsymbol{e}_k) w'_k(\boldsymbol{y}),$$

and

$$⨍_{\mathcal{Y}^*} \boldsymbol{h}_d^0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \bar{\chi}\mathcal{H}_d(\boldsymbol{u}^0) + \mathbf{H}_d\boldsymbol{u}^0,$$

where $\mathbf{H}_d$ is a $(3,3)$ symmetric matrix defined by:

$$(\mathbf{H}_d)_{ij} = \frac{1}{\bar{\chi}} \int_{\mathcal{Y}} \big(\nabla w'_i(\boldsymbol{y}) + \chi_{\mathcal{Y}^*}(\boldsymbol{y})\boldsymbol{e}_i\big)\cdot\big(\nabla w'_j(\boldsymbol{y}) + \chi_{\mathcal{Y}^*}\boldsymbol{e}_j\big)\,\mathrm{d}\boldsymbol{y} - \delta_i^j, \tag{3.1}$$

where δ_i^j is Kronecker's symbol.

Proof. Let φ^ε be a potential in $\mathrm{W}^1_0(\mathbb{R}^3)$ such that $\nabla\varphi^\varepsilon = \mathcal{H}_d(\boldsymbol{u}^\varepsilon)$. Since the sequence $\boldsymbol{u}^\varepsilon$ remains bounded in $\mathbb{L}^2(\mathbb{R}^3)$, the sequence φ^ε remains bounded in $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$. There exists φ^0 in $\mathrm{W}^1_0(\mathbb{R}^3)$ and φ^0_1 in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{H}^1_\#(\mathcal{Y}))$, such that for all ϕ in $\mathcal{C}_c(\mathbb{R}^3)\otimes\mathcal{C}^\infty_\#(\mathcal{Y})$, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\varphi^\varepsilon(\boldsymbol{x})\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3}\varphi^0(\boldsymbol{x})\left(\int_{\mathcal{Y}}\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\,\mathrm{d}\boldsymbol{x}, \tag{3.2a}$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\nabla^\varepsilon_\varphi(\boldsymbol{x})\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3}\int_{\mathcal{Y}}\left(\nabla_{\boldsymbol{x}}\varphi^0(\boldsymbol{x})+\nabla_{\boldsymbol{y}}\varphi^0_1(\boldsymbol{x},\boldsymbol{y})\right)\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}. \tag{3.2b}$$

But, for all ϕ in $\mathcal{C}^\infty_c(\mathbb{R}^3)\otimes\mathcal{C}^\infty_\#(\mathcal{Y})$,

$$\begin{aligned}\int_\Omega(\boldsymbol{u}^\varepsilon(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}})\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}+\frac{1}{\varepsilon}\int_\Omega(\boldsymbol{u}^\varepsilon(\boldsymbol{x})\cdot\nabla_{\boldsymbol{y}})\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x} = \\ = -\int_{\mathbb{R}^3}\nabla_{\boldsymbol{x}}\varphi^\varepsilon\cdot\nabla_{\boldsymbol{x}}\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}-\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\nabla_{\boldsymbol{x}}\varphi^\varepsilon\cdot\nabla_{\boldsymbol{y}}\phi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.3}$$

In (3.3), we choose ϕ independent of $\boldsymbol{y}$ and compute the limit as ε goes to 0.

$$\int_\Omega(\int_{\mathcal{Y}}\boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\cdot\nabla_{\boldsymbol{x}})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3}\nabla_{\boldsymbol{x}}\varphi^0(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0,$$

since $⨍_{\mathcal{Y}}\nabla_{\boldsymbol{y}}\varphi^0_1\,\mathrm{d}\boldsymbol{y} = 0$. Thus, $\nabla_{\boldsymbol{x}}\varphi_0 = \mathcal{H}_d(\int_{\mathcal{Y}}\boldsymbol{u}^0(\cdot,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y})$.

To compute φ^0_1, we multiply (3.3) by ε and take the two-scale limit, we have:

$$\begin{aligned}\int_\Omega\int_{\mathcal{Y}}(\boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}})\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}+ \\ +\int_{\mathbb{R}^3}\int_{\mathcal{Y}}(\nabla_{\boldsymbol{x}}\varphi^0(\boldsymbol{x})+\nabla_{\boldsymbol{y}}\varphi^0_1(\boldsymbol{x},\boldsymbol{y}))\cdot\nabla_{\boldsymbol{y}}\phi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = 0,\end{aligned}$$

for all ϕ in $\mathcal{C}^\infty_c(\mathbb{R}^3)\otimes\mathcal{C}^\infty_\#(\mathcal{Y})$. For all $\boldsymbol{x}$ in Ω, for all ψ in $\mathcal{C}^\infty_\#(\mathcal{Y})$, we have

$$\int_{\mathcal{Y}}(\boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y})+\nabla_{\boldsymbol{y}}\varphi^0_1(\boldsymbol{x},\boldsymbol{y}))\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0,$$

because

$$\int_{\mathcal{Y}}\nabla_{\boldsymbol{x}}\varphi^0(\boldsymbol{x})\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \nabla_{\boldsymbol{x}}\varphi^0(\boldsymbol{x})\cdot\int_{\partial\mathcal{Y}}\boldsymbol{\nu}\psi(\boldsymbol{y})\,\mathrm{d}\sigma(\boldsymbol{y}) = 0. \qquad \square$$

4 Application: homogenization of the Landau-Lifshitz equation in perforated domains

In this section, we homogenize the Landau-Lifshitz equation in perforated domains. Something we have already done for multilayers in [14]. The Landau-Lifshitz system models the behavior of ferromagnetic materials, see [11]. In this section, we do not restrict ourselves to an isotropic exchange interaction.

4.1 The micromagnetic model

In this subsection, we recall briefly a possible model of ferromagnetism: the micromagnetic model. The magnetic state of a ferromagnetic material is characterized by two vector fields over $\mathbb{R}^3$, the magnetization $\boldsymbol{m}$ and the excitation $\boldsymbol{h}$. The magnetization is null outside of the ferromagnetic body $\mathcal{O}$ and satisfy a non convex constraint $|\boldsymbol{m}| = 1$ inside $\mathcal{O}$.

The excitation $\boldsymbol{h}$ is given by

$$\boldsymbol{h} = \operatorname{div}(\mathbf{A}\nabla\boldsymbol{m}) + \mathbf{K}\boldsymbol{m} + \mathcal{H}_d(\boldsymbol{m}),$$

where $\mathbf{A} = (A_{i,j})_{1\leq i,j\leq 3}$ and $\mathbf{K} = (K_{i,j})_{1\leq i,j\leq 3}$ are two symmetric positive matrices field over $\mathbb{R}^3$ of class $\mathcal{C}^\infty(\overline{\Omega}) \otimes \mathcal{C}^\infty_\#(\mathcal{Y})$. We suppose that $\mathbf{A}$ is uniformly coercive: *i.e.* there exists a constant $\beta > 0$ such that, for all $(\boldsymbol{x}, \boldsymbol{y})$ in $\Omega \times \mathcal{Y}$, for all (ξ_1, ξ_2, ξ_3) in $\mathbb{R}^3$,

$$\sum_{i,j=1}^{3} A_{i,j}(\boldsymbol{x}, \boldsymbol{y})\xi_i\xi_j \geq \beta \left(\sum_{i=1}^{3} {\xi_i}^2 \right).$$

The Landau-Lifshitz equation is a phenomenological nonlinear PDE that models the evolution problem of the magnetization in a ferromagnetic material.

$$\frac{\partial \boldsymbol{m}}{\partial t} = -\boldsymbol{m} \wedge \boldsymbol{h} - \boldsymbol{m} \wedge (\boldsymbol{m} \wedge \boldsymbol{h}), \tag{4.1a}$$

It is associated with the nonconvex constraint

$$|\boldsymbol{m}| = \begin{cases} 1 & \text{in } \mathcal{O}, \\ 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{O}, \end{cases} \tag{4.1b}$$

the initial condition

$$\boldsymbol{m}(\cdot, 0) = \boldsymbol{m}_0, \tag{4.1c}$$

and the boundary condition

$$\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\mathcal{O}. \tag{4.1d}$$

We recall the rigorous definition of weak solutions to the Landau-Lifshitz system (4.1).

Definition 12. Let $\boldsymbol{m}_0^\varepsilon$ be in $\mathbb{H}^1(\Omega^\varepsilon)$ such that $|\boldsymbol{m}_0^\varepsilon| = 1$ a.e. in Ω^ε. A vector field $\boldsymbol{m}^\varepsilon$ is a weak solution to the Landau-Lifshitz system if

(1) $\boldsymbol{m}^\varepsilon$ belongs to $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega^\varepsilon))$ and to $\mathbb{H}^1(\Omega^\varepsilon \times (0,T))$, for all time $T > 0$ and $|\boldsymbol{m}^\varepsilon| = 1$ a.e. in $\Omega^\varepsilon \times \mathbb{R}^+$.

(2) $\boldsymbol{m}^\varepsilon$ satisfy

$$\begin{aligned}\iint_{\Omega^\varepsilon\times(0,T)} \frac{\partial \boldsymbol{m}^\varepsilon}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \alpha\iint_{\Omega^\varepsilon\times(0,T)}\left(\boldsymbol{m}^\varepsilon\wedge\frac{\partial \boldsymbol{m}^\varepsilon}{\partial t}\right)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = (1+\alpha^2)\iint_{\Omega^\varepsilon\times(0,T)}\sum_{i,j=1}^{3}\left(\boldsymbol{m}^\varepsilon\wedge A_{i,j}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\frac{\partial \boldsymbol{m}^\varepsilon}{\partial x_i}\right)\cdot\frac{\partial\boldsymbol{\phi}}{\partial x_j}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + (1+\alpha^2)\iint_{\Omega^\varepsilon\times(0,T)}\left(\boldsymbol{m}^\varepsilon\wedge\mathbf{K}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\boldsymbol{m}^\varepsilon\right)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ - (1+\alpha^2)\iint_{\Omega^\varepsilon\times(0,T)}\left(\boldsymbol{m}^\varepsilon\wedge\mathcal{H}_d(\boldsymbol{m}^\varepsilon)\right)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \end{aligned} \tag{4.2a}$$

for all $\boldsymbol{\phi}$ in $\mathcal{C}^\infty(\overline{\Omega\times(0,T)};\mathbb{R}^3)$.

(3) In the sense of traces:

$$\boldsymbol{m}^\varepsilon(\cdot,0) = \boldsymbol{m}_0^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{4.2b}$$

(4) $\boldsymbol{m}^\varepsilon$ satisfies the energy inequality

$$\mathrm{E}^\varepsilon(\boldsymbol{m}^\varepsilon(T)) + \frac{\alpha}{1+\alpha^2}\int_0^T \left\|\frac{\partial \boldsymbol{m}^\varepsilon}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega^\varepsilon)}\mathrm{d}t \le \mathrm{E}^\varepsilon(\boldsymbol{m}_0^\varepsilon). \tag{4.2c}$$

where, for all $\boldsymbol{u}$ in $\mathbb{H}^1(\Omega^\varepsilon)$,

$$\begin{aligned}\mathrm{E}^\varepsilon(\boldsymbol{u}) = \frac{1}{2}\int_{\Omega^\varepsilon}\sum_{i,j=1}^{3}A_{i,j}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\frac{\partial \boldsymbol{u}}{\partial x_i}\cdot\frac{\partial \boldsymbol{u}}{\partial x_j}\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\Omega^\varepsilon}\boldsymbol{u}\cdot\mathbf{K}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \\ + \frac{1}{2}\int_{\mathbb{R}^3}|\mathcal{H}_d(\boldsymbol{u})|^2\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

We recall the following result:

Theorem 13. *Let $\boldsymbol{m}_0^\varepsilon$ be in $\mathbb{H}^1(\Omega^\varepsilon)$ such that $|\boldsymbol{m}_0^\varepsilon| = 1$ a.e. in Ω^ε. Then, there exists a weak solution to the Landau-Lifshitz system in the sense of Definition 12.*

Proof. For the proof of existence of weak solutions, one can consult Alouges-Soyeur [2] for isotropic exchange, *i.e.* $\mathbf{A} = A\mathrm{I}_3$. See Hamdache-Tilioua [13] for the generalization of the proof of existence of solutions when exchange is anisotropic. $\square$

4.2 The homogenized system

In this section, we homogenize the Landau-Lifshitz system (4.2) in perforated domains via the two-scale convergence method:

Theorem 14. *Let $(\boldsymbol{m}_0^\varepsilon)_\varepsilon$ be a sequence in $\mathbb{H}^1(\Omega^\varepsilon)$, $|\boldsymbol{m}_0^\varepsilon| = 1$ a.e. in Ω^ε such that $\|\boldsymbol{m}_0^\varepsilon\|_{\mathbb{H}^1(\Omega^\varepsilon)}$ remains bounded independently of ε. By $\bar{\boldsymbol{m}}_0^\varepsilon$, we denote the extension by 0 of $\boldsymbol{m}^\varepsilon$ outside Ω^ε. We suppose there exists $\boldsymbol{m}_0^0$ in $\mathbb{H}^1(\Omega)$ such that*

$$\lim_{\varepsilon \to 0} \bar{\boldsymbol{m}}_0^\varepsilon = \bar{\chi} \boldsymbol{m}_0^0 \quad \textit{weakly in } \mathrm{L}^2(\Omega). \tag{4.3}$$

For all ε, we let $\boldsymbol{m}^\varepsilon$ be one weak solution of system (4.2) with $\boldsymbol{m}_0^\varepsilon$ as initial condition.

Then, the $\mathbb{H}^1(\Omega^\varepsilon \times (0,T))$ norm of $\boldsymbol{m}^\varepsilon$ remains bounded independently of ε. There exists, for all $\varepsilon > 0$, an extension of $\boldsymbol{m}^\varepsilon$ in $\mathbb{H}^1(\Omega)$ denoted by $\widetilde{\boldsymbol{m}}^\varepsilon$ such that the sequence $(\widetilde{\boldsymbol{m}}^\varepsilon)_{\{\varepsilon>0\}}$ remains bounded in $\mathbb{H}^1(\Omega \times (0,T))$.

Modulo a subsequence, $\widetilde{\boldsymbol{m}}^\varepsilon$ two-scale converges to $\widetilde{\boldsymbol{m}}^0$ in $\mathbb{H}^1(\Omega \times (0,T))$. Any limit $\widetilde{\boldsymbol{m}}^0$ belongs to $\mathrm{L}^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ and to $\mathrm{H}^1(0,T; \mathbb{L}^2(\Omega))$, for all time $T > 0$ and satisfy equation (4.9), i.e. formally:

$$\frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial t} - \alpha \widetilde{\boldsymbol{m}}^0 \wedge \frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial t} = -(1+\alpha^2) \widetilde{\boldsymbol{m}}^0 \wedge \Big(\mathrm{div}((\mathbf{A}^* \cdot \nabla) \widetilde{\boldsymbol{m}}^0) - \bar{\mathbf{K}} \widetilde{\boldsymbol{m}}^0$$
$$+ \bar{\chi} \mathcal{H}_d(\widetilde{\boldsymbol{m}}^0) + \mathbf{H}_d \widetilde{\boldsymbol{m}}^0 \Big),$$

in $\Omega \times \mathbb{R}^+$,

$$\widetilde{\boldsymbol{m}}^0(\cdot, 0) = \boldsymbol{m}_0^0 \qquad \textit{in } \Omega,$$
$$|\widetilde{\boldsymbol{m}}^0| = 1 \qquad \textit{in } \Omega \times \mathbb{R}^+,$$

and the Neumann boundary conditions

$$\frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial \boldsymbol{\nu}} = 0 \qquad \textit{in } \partial\Omega \times \mathbb{R}^+,$$

where $\mathbf{H}_d$ is defined by (3.1), $\bar{\mathbf{K}} = \fint_{\mathcal{Y}^*} \mathbf{K}(\boldsymbol{x}, \boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}$ and $\mathbf{A}^*$ is the usual homogenized operator for elliptic operators, see Lemma 19.

Before proving the theorem, we make the following remark

Remark 15. Since all known uniqueness results for weak solutions to the Landau-Lifshitz system are negative, there is no reason for the whole sequence $\widetilde{\boldsymbol{m}}^\varepsilon$ to converge in Theorem 14.

By the energy inequality, the $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega^\varepsilon))$ norm of $\boldsymbol{m}^\varepsilon$ and the $\mathbb{L}^2(\Omega^\varepsilon\times\mathbb{R}^+)$ norm of $\frac{\partial \boldsymbol{m}^\varepsilon}{\partial t}$ remain bounded. We set $\widetilde{\boldsymbol{m}}^\varepsilon = P_\varepsilon(\boldsymbol{m}^\varepsilon)$, where P_ε is the extension operator provided by Lemma 8. The vector field $\widetilde{\boldsymbol{m}}^\varepsilon$ belongs to $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ and satisfy:

$$\widetilde{\boldsymbol{m}}^\varepsilon = \boldsymbol{m}^\varepsilon \qquad \text{in } \Omega^\varepsilon\times\mathbb{R}^+, \qquad\qquad \left\| \frac{\partial \widetilde{\boldsymbol{m}}^\varepsilon}{\partial t} \right\|_{\mathbb{L}^2(\Omega\times(0,T))} \le C.$$

According to Proposition 1.14 of [1], there exists $\widetilde{\boldsymbol{m}}^0$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))$ such that $\frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial t}$ belongs to $\mathbb{L}^2(\Omega\times(0,T))$, and $\widetilde{\boldsymbol{m}}_1^0$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^2(\Omega;\mathbb{H}^1_\#(\mathcal{Y})))$ such that, up to a subsequence,

- $\widetilde{\boldsymbol{m}}^\varepsilon$ two-scale converges to $\widetilde{\boldsymbol{m}}^0$.
- $\widetilde{\boldsymbol{m}}^\varepsilon$ strongly converges to $\widetilde{\boldsymbol{m}}^0$ in $\mathbb{L}^2(\Omega)$.
- $\nabla\widetilde{\boldsymbol{m}}^\varepsilon$ two-scales converges to $\nabla_{\boldsymbol{x}}\widetilde{\boldsymbol{m}}^0 + \nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{m}}_1^0$.
- $\frac{\partial \widetilde{\boldsymbol{m}}^\varepsilon}{\partial t}$ two-scales converges to $\frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial t}$.

Our goal is establishing the system that must be satisfied by $\widetilde{\boldsymbol{m}}^0$.

As the canonical injection from $\mathbb{H}^1(\Omega\times(0,T))$ into $\mathbb{L}^2(\Omega\times(0,T))$ is compact, we can take the two-scale limit in nonlinear terms by Theorem 4. Moreover, for a subsequence we have convergence a.e., $|\widetilde{\boldsymbol{m}}^0| = 1$ a.e. in $\Omega\times\mathbb{R}^+$.

We begin by computing the initial condition

Lemma 16. *The trace $\widetilde{\boldsymbol{m}}^0$ at instant $t=0$ is $\boldsymbol{m}_0^0$, i.e.*

$$\widetilde{\boldsymbol{m}}^0(\cdot,0) = \boldsymbol{m}_0^0.$$

Proof. Let $\widetilde{\boldsymbol{m}}_0^0$ be the two-scale limit of $\widetilde{\boldsymbol{m}}_0^\varepsilon$. This limit does not depend on the fast variable of $\boldsymbol{y}$ because $\widetilde{\boldsymbol{m}}_0^\varepsilon$ remains bounded in $\mathbb{H}^1(\Omega)$. The weak limit in $\mathbb{L}^2(\Omega)$ of $\bar{\boldsymbol{m}}^\varepsilon$ is thus $\bar{\chi}\widetilde{\boldsymbol{m}}^0$. We conclude by hypothesis (4.3). ☐

Then, we prove a simple lemma:

Lemma 17. *For all integers i, $1\le i\le 3$, $\frac{\partial \widetilde{\boldsymbol{m}}_1^0}{\partial y_i}$ is orthogonal to $\widetilde{\boldsymbol{m}}^0$ almost everywhere on $\Omega\times\mathbb{R}^+$.*

Proof. We have $|\widetilde{\boldsymbol{m}}^\varepsilon| = 1$. Since $\widetilde{\boldsymbol{m}}^0$ is the strong limit of $\widetilde{\boldsymbol{m}}^\varepsilon$, we have $|\widetilde{\boldsymbol{m}}^0| = 1$. We can take the two-scale limit in expression

$$\lim_{\varepsilon\to 0} \iint_{\Omega\times(0,T)} \chi_\varepsilon(\boldsymbol{x})\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\cdot\frac{\partial\widetilde{\boldsymbol{m}}^\varepsilon}{\partial x_i}(\boldsymbol{x},t)\phi\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= \iint_{\Omega\times(0,T)} \widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t)\cdot\int_{\mathcal{Y}^*}\left(\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},t)+\frac{\partial\widetilde{\boldsymbol{m}}^0_1}{\partial y_i}(\boldsymbol{x},t,\boldsymbol{y})\right)\phi(\boldsymbol{x},t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0,$$

for all integers i, $1\le i\le 3$, and all ϕ in $\mathcal{C}^\infty(\overline{\Omega\times(0,T)})\otimes\mathcal{C}^\infty_\#(\mathcal{Y})$. Therefore, $\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_i}+\frac{\partial\widetilde{\boldsymbol{m}}^0_1}{\partial y_i}$ is orthogonal a.e. in $\Omega\times\mathbb{R}^+\times\mathcal{Y}^*$ to $\widetilde{\boldsymbol{m}}^0$. Since $|\widetilde{\boldsymbol{m}}^0| = 1$, $\frac{\partial\widetilde{\boldsymbol{m}}^0_1}{\partial y_i}$ is also orthogonal a.e. in $\Omega\times\mathcal{Y}^*$ to $\widetilde{\boldsymbol{m}}^0$. □

We may now compute the limit of equation (4.2a). We have:

$$\begin{aligned}
&\iint_{\Omega^\varepsilon\times(0,T)} \frac{\partial\widetilde{\boldsymbol{m}}^\varepsilon}{\partial t}(\boldsymbol{x},t)\cdot\boldsymbol{\phi}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&-\alpha\iint_{\Omega^\varepsilon\times(0,T)}\left(\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\wedge\frac{\partial\widetilde{\boldsymbol{m}}^\varepsilon}{\partial t}(\boldsymbol{x},t)\right)\cdot\boldsymbol{\phi}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&=(1+\alpha^2)\iint_{\Omega^\varepsilon\times(0,T)}\sum_{i,j=1}^3 A_{i,j}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\left(\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\wedge\frac{\partial\widetilde{\boldsymbol{m}}^\varepsilon}{\partial x_i}(\boldsymbol{x},t)\right)\cdot\frac{\partial\boldsymbol{\phi}}{\partial x_j}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\frac{(1+\alpha^2)}{\varepsilon}\sum_{i,j=1}^3\iint_{\Omega^\varepsilon\times(0,T)} A_{i,j}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\left(\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\wedge\frac{\partial\widetilde{\boldsymbol{m}}^\varepsilon}{\partial x_i}(\boldsymbol{x},t)\right)\cdot\frac{\partial\boldsymbol{\phi}}{\partial y_j}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+(1+\alpha^2)\iint_{\Omega^\varepsilon\times(0,T)}\left(\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\wedge\mathbf{K}\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\right)\cdot\boldsymbol{\phi}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\iint_{\Omega^\varepsilon\times(0,T)}(\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{x},t)\wedge\mathcal{H}_d(\chi_\varepsilon\widetilde{\boldsymbol{m}}^\varepsilon)(\boldsymbol{x},t))\cdot\boldsymbol{\phi}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,
\end{aligned}\tag{4.5}$$

for all $\boldsymbol{\phi}$ in $\mathcal{C}^\infty(\overline{\Omega\times(0,T)})\otimes\mathcal{C}^\infty_\#(\mathcal{Y};\mathbb{R}^3)$.

We begin by computing the limit of the demagnetization field term. By Proposition 11, the weak $\mathbb{L}^2(\mathbb{R}^3)$ limit of $\chi_\varepsilon\mathcal{H}_d(\chi_\varepsilon\widetilde{\boldsymbol{m}}^\varepsilon)$ is

$$\bar\chi\fint_{\mathcal{Y}^*}\boldsymbol{h}^0_d(\boldsymbol{x},t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \bar\chi\big(\bar\chi\mathcal{H}_d(\widetilde{\boldsymbol{m}}^0)+\mathbf{H}_d\widetilde{\boldsymbol{m}}^0\big),$$

where

$$(\mathbf{H}_d)_{ij} = \frac{1}{\bar\chi}\int_{\mathcal{Y}}\big(\nabla w'_i(\boldsymbol{y})+\chi_{\mathcal{Y}^*}(\boldsymbol{y})\boldsymbol{e}_i\big)\cdot\big(\nabla w'_j(\boldsymbol{y})+\chi_{\mathcal{Y}^*}\boldsymbol{e}_j\big)\,\mathrm{d}\boldsymbol{y}-\delta^j_i.$$

Before studying the exchange term, that will converges to the usual homogenized limit for elliptic operators, we introduce as in Bensoussan et al [4] some special functions.

Definition 18. For any integer i, $1 \leq i \leq 3$, let w_i in $\mathrm{H}^1_\#(\mathcal{Y}^*)$, be the unique solution to

$$\int_{\mathcal{Y}^*} \mathbf{A}(\boldsymbol{x},\boldsymbol{y})\nabla_{\boldsymbol{y}}\boldsymbol{\psi}\cdot(\nabla_{\boldsymbol{y}}w_i(\boldsymbol{x},\boldsymbol{y})+\boldsymbol{e}_i)\,\mathrm{d}\boldsymbol{y}=0, \qquad \int_{\mathcal{Y}^*} w_i\,\mathrm{d}\boldsymbol{y}=0,$$

for all $\boldsymbol{x}$ in Ω, and $\boldsymbol{\psi}$ in $\mathrm{H}^1_\#(\mathcal{Y}^*)$, $\boldsymbol{e}_i$ being the ith vector of the canonical basis of $\mathbb{R}^3$. By $\boldsymbol{w}$, we denote the horizontal vector $[w_1, w_2, w_3]$.

Concerning the exchange term, we have the following lemma:

Lemma 19. *For all $\boldsymbol{x}$ in Ω:*

$$\sum_{i=1}^{3}\int_{\mathcal{Y}^*} A_{i,j}\left(\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},\boldsymbol{y})+\frac{\partial\widetilde{\boldsymbol{m}}^0_1}{\partial y_i}(\boldsymbol{x},\boldsymbol{y})\right)\mathrm{d}\boldsymbol{y}=\bar{\chi}\sum_{i=1}^{3}A^*_{i,j}\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},\boldsymbol{y}), \quad (4.6)$$

where

$$A^*_{i,j}=\int_{\mathcal{Y}^*}\mathbf{A}(\boldsymbol{e}_j+\nabla w_j(\boldsymbol{x},\boldsymbol{y}))\cdot(\boldsymbol{e}_i+\nabla w_i(\boldsymbol{x},\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}. \quad (4.7)$$

Proof. We multiply equation (4.5) by ε and take the limit as ε goes to 0. Only one term converges to a nonzero limit. Thus,

$$\sum_{i,j=1}^{3}\iint_{\Omega\times(0,T)}\int_{\mathcal{Y}^*}A_{i,j}(\boldsymbol{x},t,\boldsymbol{y})\left(\widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t)\wedge\left(\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},t)+\frac{\partial\widetilde{\boldsymbol{m}}^0_1}{\partial y_i}(\boldsymbol{x},t,\boldsymbol{y})\right)\right)\cdot\frac{\partial\boldsymbol{\phi}}{\partial y_j}(\boldsymbol{x},t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0, \quad (4.8)$$

for all $\boldsymbol{\phi}$ in $\mathcal{C}^\infty(\overline{\Omega\times(0,T)})\otimes\mathcal{C}^\infty_\#(\mathcal{Y};\mathbb{R}^3)$. By Lemma 17, for all $\boldsymbol{x},t$ in $\Omega\times\mathbb{R}^+$, and all $\boldsymbol{\psi}$ in $\mathcal{C}^\infty_\#(\mathcal{Y};\mathbb{R}^3)$,

$$\sum_{i,j=1}^{3}\int_{\mathcal{Y}^*}A_{i,j}(\boldsymbol{x},t,\boldsymbol{y})\left(\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},t)+\frac{\partial\widetilde{\boldsymbol{m}}^0_1}{\partial y_i}(\boldsymbol{x},t,\boldsymbol{y})\right)\cdot\frac{\partial\boldsymbol{\psi}}{\partial y_j}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}=0.$$

Therefore,

$$\widetilde{\boldsymbol{m}}^0_1(\boldsymbol{x},t,\boldsymbol{y})=\sum_{k=1}^{3}\frac{\partial\widetilde{\boldsymbol{m}}^0}{\partial x_k}(\boldsymbol{x},t)w_k(\boldsymbol{x},\boldsymbol{y}).$$

Thus, for all integers j, $1 \leq j \leq 3$,

$$\begin{aligned}
&\sum_{i=1}^{3} \int_{\mathcal{Y}^*} A_{i,j}(\boldsymbol{x},\boldsymbol{y}) \left(\frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},t) + \frac{\partial \widetilde{\boldsymbol{m}}_1^0}{\partial y_i}(\boldsymbol{x},t,\boldsymbol{y}) \right) \mathrm{d}\boldsymbol{y} \\
&\quad = \left(\sum_{i=1}^{3} \int_{\mathcal{Y}^*} \left(A_{i,j}(\boldsymbol{x},\boldsymbol{y}) + \sum_{k=1}^{3} A_{k,j}(\boldsymbol{x},\boldsymbol{y}) \frac{\partial w_i}{\partial y_k}(\boldsymbol{x},\boldsymbol{y}) \right) \mathrm{d}\boldsymbol{y} \right) \frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A^*_{i,j} &= \int_{\mathcal{Y}^*} A_{i,j}(\boldsymbol{x},\boldsymbol{y}) + \sum_{k=1}^{3} A_{k,j}(\boldsymbol{x},\boldsymbol{y}) \frac{\partial w_i}{\partial y_k}(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathcal{Y}^*} \mathbf{A}\boldsymbol{e}_j \cdot (\boldsymbol{e}_i + \nabla w_i(\boldsymbol{x},\boldsymbol{y})) \,\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathcal{Y}^*} \mathbf{A}(\boldsymbol{e}_j + \nabla w_j(\boldsymbol{x},\boldsymbol{y})) \cdot (\boldsymbol{e}_i + \nabla w_i(\boldsymbol{x},\boldsymbol{y})) \,\mathrm{d}\boldsymbol{y} \\
&= \bar{\chi} ⨍_{\mathcal{Y}^*} \mathbf{A}(\boldsymbol{e}_j + \nabla w_j(\boldsymbol{x},\boldsymbol{y})) \cdot (\boldsymbol{e}_i + \nabla w_i(\boldsymbol{x},\boldsymbol{y})) \,\mathrm{d}\boldsymbol{y}. \qquad \square
\end{aligned}$$

Looking in the literature concerning homogenization[1], we notice that:

Remark 20. The homogenized exchange operator is the same as the classical homogenized operator obtained by homogenization of the elliptic equation associated to this operator.

We may now take the limit in equation (4.5) when $\boldsymbol{\phi}$ is independent of $\boldsymbol{y}$. For all $\boldsymbol{\phi}$ in $\mathcal{C}^\infty(\overline{\Omega \times (0,T)}; \mathbb{R}^3)$:

$$\begin{aligned}
&\iint_{\Omega\times(0,T)} \left(\frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial t}(\boldsymbol{x},t) - \widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t) \wedge \frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial t}(\boldsymbol{x},t) \right) \cdot \boldsymbol{\phi}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= (1+\alpha^2) \iint_{\Omega\times(0,T)} \sum_{i,j=1}^{3} A^*_{i,j}(\boldsymbol{x}) \left(\widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t) \wedge \frac{\partial \widetilde{\boldsymbol{m}}^0}{\partial x_i}(\boldsymbol{x},t) \right) \cdot \frac{\partial \boldsymbol{\phi}}{\partial x_j}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + (1+\alpha^2) \iint_{\Omega\times(0,T)} \left(\widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t) \wedge \left(\bar{\mathbf{K}}(\boldsymbol{x}) \widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t) \right) \right) \cdot \boldsymbol{\phi}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&-(1+\alpha^2) \iint_{\Omega\times(0,T)} \left(\widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t) \wedge \left(\bar{\chi}\mathcal{H}_d(\widetilde{\boldsymbol{m}}^0)(\boldsymbol{x},t) + \mathbf{H}_d \widetilde{\boldsymbol{m}}^0(\boldsymbol{x},t) \right) \right) \cdot \boldsymbol{\phi}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned} \tag{4.9}$$

[1]See [4], for the case without holes, [6] for the homogenization with holes and section 2 of [1] for the homogenization of elliptic operators using two-scale convergence

5 Conclusion

We have computed the homogenized demagnetization field operator in perforated domains via the two-scale convergence method. We have then used the result to homogenize the Landau-Lifshitz system in domains periodically perforated by regular homothetic holes. The reverse problem of homogenizing the Landau-Lifshitz equation in a nonconnex periodic domain by having the ferromagnetic material fill $\Omega \setminus \Omega^\varepsilon$ instead of Ω^ε has many real world applications. However, this problem is much more mathematically challenging: the nonexistence a good sequence of extension operators in such a case make homogenization of nonlinear equations much more difficult.

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